Journal of Computational and Applied Mechanics, Vol. 4., No. 1., (2003), pp. 71–88

STABILIZATION PARAMETERS IN SUPG AND PSPG FORMULATIONS

Tayfun Tezduyar and Sunil Sathe
Mechanical Engineering, Rice University – MS 321
6100 Main Street, Houston, Texas 77005, USA
tezduyar@mems.rice.edu

[Received: June 29, 2002]

Abstract. We describe how we determine the stabilization parameters and element length scales used in the stabilized finite element formulations in fluid mechanics. These formulations include the interface-tracking and interface-capturing techniques we developed for computation of flow problems with moving boundaries and interfaces. The stabilized formulations we focus on are the streamline-upwind/Petrov-Galerkin (SUPG) and pressure-stabilizing/Petrov-Galerkin (PSPG) methods. The stabilization parameters described here are designed for the semi-discrete and space-time formulations of the advection-diffusion equation and the Navier-Stokes equations of incompressible flows.

Mathematical Subject Classification: 76M10
Keywords: stabilization parameters, element length scales, SUPG formulation, PSPG formulation, space-time formulation

1. Introduction

Most finite element techniques and computations reported in recent literature for computational fluid mechanics are based on stabilized formulations. The interface-tracking and interface-capturing techniques we have developed in recent years (see [1, 2, 3, 4]) for flows with moving boundaries and interfaces are also based on stabilized formulations. An interface-tracking technique, such as the Deforming-Spatial-Domain/Stabilized Space-Time (DSD/SST) formulation [1], requires meshes that "track" the interfaces. The mesh needs to be updated as the flow evolves. In interface-capturing techniques, such as one designed for two-fluid flows, the computations are based on spatial domains that are typically not moving or deforming. An interface function, marking the location of the interface, needs to be computed to "capture" the interface over the non-moving mesh.

The finite element flow techniques we have developed, including the interface-tracking and interface-capturing techniques, are based on the streamline-upwind/Petrov-Galerkin (SUPG) [5], Galerkin/least-squares (GLS) [6], and pressure-stabilizing/Petrov-Galerkin (PSPG) [1] formulations. In the interface-capturing techniques, stabilized semi-discrete formulations are used for both the Navier-Stokes equations of incompressible flows and the advection equation governing the time-evolution of an interface function marking the interface location. These stabilization techniques prevent numerical oscillations and other instabilities in solving problems with advection-dominated flows and when using equal-order interpolation functions for velocity and pressure. In these stabilized formulations, judicious selection of the stabilization parameter, which is almost always known as "τ", plays an important role in determining the accuracy of the formulation. This stabilization parameter involves a measure of the local length scale (also known as "element length") and other parameters such as the local Reynolds and Courant numbers. Various element lengths and τs were proposed starting with those in [5] and [7], followed by the one introduced in [8], and those proposed in the subsequently reported SUPG, GLS and PSPG methods. A number of τs, dependent upon spatial and temporal discretizations, were introduced and tested in [9]. More recently, τs which are applicable to higher-order elements were proposed in [10].

Ways to calculate τs from the element-level matrices and vectors were first introduced in [11]. These new definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They take into account the local length scales, advection field and the element-level Reynolds number. Based on these definitions, a τ can be calculated for each element, or even for each element node or degree of freedom or element equation. Certain variations and complements of these new τs were introduced in [12, 4, 13]. In this paper, we describe the element-matrix-based and element-vector-based τs designed for the semi-discrete and space-time formulations of the advection-diffusion equation and the Navier-Stokes equations of incompressible flows. We also describe approximate versions of these τs that are based on the local length scales for the advection- and diffusion-dominated limits. In the test computations reported in Section 13, the performance of the stabilization parameters is evaluated for the advection-diffusion and Navier-Stokes equations.

2. Governing Equations

Let $\Omega_t \subset I\!R^{n_{sd}}$ be the spatial fluid mechanics domain with boundary Γ_t at time $t \in (0, T)$, where the subscript t indicates the time-dependence of the spatial domain. The Navier-Stokes equations of incompressible flows can be written on Ω_t and $\forall t \in (0, T)$ as

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{f}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0, \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2.2}$$

where ρ, $\mathbf{u}$ and $\mathbf{f}$ are the density, velocity and the external force, and $\boldsymbol{\sigma}$ is the stress tensor:

$$\boldsymbol{\sigma}(p, \mathbf{u}) = -p\mathbf{I} + 2\mu\boldsymbol{\epsilon}(\mathbf{u}), \quad \boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^T). \tag{2.3}$$

Here p is pressure, $\mathbf{I}$ is the identity tensor, $\mu = \rho\nu$ is viscosity, ν is the kinematic viscosity, and $\boldsymbol{\epsilon}(\mathbf{u})$ is the strain-rate tensor. The essential and natural boundary conditions for equation (2.1) are represented as

$$\mathbf{u} = \mathbf{g} \text{ on } (\Gamma_\mathrm{t})_\mathrm{g}, \ \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{h} \text{ on } (\Gamma_\mathrm{t})_\mathrm{h}, \tag{2.4}$$

where $(\Gamma_t)_g$ and $(\Gamma_t)_h$ are complementary subsets of the boundary Γ_t, $\mathbf{n}$ is the unit normal vector, and $\mathbf{g}$ and $\mathbf{h}$ are given functions. A divergence-free velocity field $\mathbf{u}_0(\mathbf{x})$ is specified as the initial condition.

If the problem does not involve any moving boundaries or interfaces, the spatial domain does not need to change with respect to time, and the subscript t can be dropped from Ω_t and Γ_t. This might be the case even for flows with moving boundaries and interfaces, if in the formulation used the spatial domain is not defined to be the part of the space occupied by the fluid(s). For example, we can have a fixed spatial domain, and model the fluid-fluid interfaces by assuming that the domain is occupied by two immiscible fluids, A and B, with densities ρ_A and ρ_B and viscosities μ_A and μ_B. In modeling a free-surface problem where Fluid B is irrelevant, we assign a sufficiently low density to Fluid B. An interface function ϕ serves as a marker identifying Fluid A and B with the definition $\phi = \{1$ for Fluid A and 0 for Fluid B$\}$. The interface between the two fluids is approximated to be at $\phi = 0.5$. In this context, ρ and μ are defined as

$$\rho = \phi\rho_A + (1-\phi)\rho_B, \;\; \mu = \phi\mu_A + (1-\phi)\mu_B. \tag{2.5}$$

The evolution of the interface function ϕ, and therefore the motion of the interface, is governed by a time-dependent advection equation, written on Ω and $\forall t \in (0,T)$ as

$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\phi = 0. \tag{2.6}$$

As a generalization of equation (2.6), let us consider over a domain Ω with boundary Γ the following time-dependent advection-diffusion equation, written on Ω and $\forall t \in (0,T)$ as

$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\phi) = 0, \tag{2.7}$$

where ϕ represents the quantity being transported (e.g., temperature, concentration), and ν is the diffusivity. The essential and natural boundary conditions associated with equation (2.7) are represented as

$$\phi = \mathrm{g} \text{ on } \Gamma_{\mathrm{g}}, \;\; \mathbf{n}\cdot\nu\boldsymbol{\nabla}\phi = \mathrm{h} \text{ on } \Gamma_{\mathrm{h}}. \tag{2.8}$$

A function $\phi_0(\mathbf{x})$ is specified as the initial condition.

3. **Stabilized Formulation for Advection-Diffusion Equation**

Let us assume that we have constructed some suitably-defined finite-dimensional trial solution and test function spaces $\mathcal{S}^h_\phi$ and $\mathcal{V}^h_\phi$. The stabilized finite element formulation of equation (2.7) can then be written as follows: find $\phi^h \in \mathcal{S}^h_\phi$ such that $\forall w^h \in \mathcal{V}^h_\phi$:

$$\begin{aligned} &\int_\Omega w^h\left(\frac{\partial\phi^h}{\partial t} + \mathbf{u}^h\cdot\boldsymbol{\nabla}\phi^h\right)d\Omega + \int_\Omega \boldsymbol{\nabla}w^h\cdot\nu\boldsymbol{\nabla}\phi^h d\Omega - \int_{\Gamma_\mathrm{h}} w^h \mathrm{h}^h d\Gamma \\ + &\sum_{e=1}^{n_{el}}\int_{\Omega^e}\tau_{\mathrm{SUPG}}\mathbf{u}^h\cdot\boldsymbol{\nabla}w^h\left(\frac{\partial\phi^h}{\partial t} + \mathbf{u}^h\cdot\boldsymbol{\nabla}\phi^h - \boldsymbol{\nabla}\cdot\left(\nu\boldsymbol{\nabla}\phi^h\right)\right)d\Omega = 0. \end{aligned} \tag{3.1}$$

Here n_{el} is the number of elements, Ω^e is the element domain, and τ_{SUPG} is the SUPG stabilization parameter.

4. Element-Matrix-Based Stabilization Parameters for Advection-Diffusion Equation

Let us use the notation $\mathbf{b} : \int_{\Omega^e}(\ldots)d\Omega : \mathbf{b}_{\mathrm{V}}$ to denote the element-level matrix $\mathbf{b}$ and element-level vector $\mathbf{b}_{\mathrm{V}}$ corresponding to the element-level integration term $\int_{\Omega^e}(\ldots)d\Omega$. We define the following element-level matrices and vectors:

$$\mathbf{m} : \quad \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \mathbf{m}_{\mathrm{V}}, \tag{4.1}$$

$$\mathbf{c} : \quad \int_{\Omega^e} w^h \mathbf{u}^h \cdot \boldsymbol{\nabla}\phi^h d\Omega \quad : \mathbf{c}_{\mathrm{V}}, \tag{4.2}$$

$$\mathbf{k} : \quad \int_{\Omega^e} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla}\phi^h d\Omega \quad : \mathbf{k}_{\mathrm{V}}, \tag{4.3}$$

$$\tilde{\mathbf{k}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \; \mathbf{u}^h \cdot \boldsymbol{\nabla}\phi^h d\Omega \quad : \tilde{\mathbf{k}}_{\mathrm{V}}, \tag{4.4}$$

$$\tilde{\mathbf{c}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_{\mathrm{V}}. \tag{4.5}$$

We define the element-level Reynolds and Courant numbers as follows:

$$Re = \frac{\|\mathbf{u}^h\|^2}{\nu} \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}, \tag{4.6}$$

$$Cr_u = \frac{\Delta t}{2} \frac{\|\mathbf{c}\|}{\|\mathbf{m}\|}, \tag{4.7}$$

$$Cr_\nu = \frac{\Delta t}{2} \frac{\|\mathbf{k}\|}{\|\mathbf{m}\|}, \tag{4.8}$$

where $\|\mathbf{b}\|$ is the norm of matrix $\mathbf{b}$.

The components of element-matrix-based τ_{SUPG} are defined as follows:

$$\tau_{\mathrm{S1}} = \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|},$$

$$\tau_{\mathrm{S2}} = \frac{\Delta t}{2} \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{c}}\|}, \tag{4.9}$$

$$\tau_{\mathrm{S3}} = \tau_{\mathrm{S1}} Re = \left(\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|} \right) Re. \tag{4.10}$$

To construct τ_{SUPG} from its components we proposed in [11] the form

$$\tau_{\mathrm{SUPG}} = \left(\frac{1}{\tau_{\mathrm{S1}}^r} + \frac{1}{\tau_{\mathrm{S2}}^r} + \frac{1}{\tau_{\mathrm{S3}}^r} \right)^{-\frac{1}{r}}. \tag{4.11}$$

The components of the element-vector-based $\tau_{\rm SUPG}$ are defined as follows:

$$\tau_{\rm SV1} = \frac{\|\mathbf{c}_{\rm V}\|}{\|\tilde{\mathbf{k}}_{\rm V}\|}, \tag{4.12}$$

$$\tau_{\rm SV2} = \frac{\|\mathbf{c}_{\rm V}\|}{\|\tilde{\mathbf{c}}_{\rm V}\|}, \tag{4.13}$$

$$\tau_{\rm SV3} = \tau_{\rm SV1} Re = \left(\frac{\|\mathbf{c}_{\rm V}\|}{\|\tilde{\mathbf{k}}_{\rm V}\|}\right) Re. \tag{4.14}$$

With these three components,

$$(\tau_{\rm SUPG})_{\rm V} = \left(\frac{1}{\tau^r_{\rm SV1}} + \frac{1}{\tau^r_{\rm SV2}} + \frac{1}{\tau^r_{\rm SV3}}\right)^{-\frac{1}{r}}. \tag{4.15}$$

5. Stabilized Formulation for Navier-Stokes Equations

Let us assume that we have some suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure: $\mathcal{S}^h_{\mathbf{u}}$, $\mathcal{V}^h_{\mathbf{u}}$, $\mathcal{S}^h_p$ and $\mathcal{V}^h_p = \mathcal{S}^h_p$. The stabilized finite element formulation of equations (2.1)-(2.2) can then be written as follows: find $\mathbf{u}^h \in \mathcal{S}^h_{\mathbf{u}}$ and $p^h \in \mathcal{S}^h_p$ such that $\forall \mathbf{w}^h \in \mathcal{V}^h_{\mathbf{u}}$ and $q^h \in \mathcal{V}^h_p$:

$$\begin{aligned}
& \int_\Omega \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}\right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) d\Omega - \int_{\Gamma_{\rm h}} \mathbf{w}^h \cdot \mathbf{h}^h d\Gamma \\
+ & \int_\Omega q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \frac{1}{\rho} [\tau_{\rm SUPG} \rho \mathbf{u}^h \cdot \nabla \mathbf{w}^h + \tau_{\rm PSPG} \nabla q^h] \cdot \\
& \left[\rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h\right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(p^h, \mathbf{u}^h) - \rho \mathbf{f}\right] d\Omega \\
+ & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\rm LSIC} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega = 0.
\end{aligned} \tag{5.1}$$

Here $\tau_{\rm PSPG}$ and $\tau_{\rm LSIC}$ are the PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters.

6. Element-Matrix-Based Stabilization Parameters for Navier-Stokes Equation

We define the following element-level matrices and vectors:

$$\mathbf{m} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \mathbf{m}_{\rm V}, \tag{6.1}$$

$$\mathbf{c} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \mathbf{c}_{\rm V}, \tag{6.2}$$

$$\mathbf{k}: \quad \int_{\Omega^e} \boldsymbol{\varepsilon}(\mathbf{w}^h) : 2\mu\boldsymbol{\varepsilon}(\mathbf{u}^h) d\Omega \quad : \mathbf{k}_{\mathrm{V}}, \tag{6.3}$$

$$\mathbf{g}: \quad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) p^h d\Omega \quad : \mathbf{g}_{\mathrm{V}}, \tag{6.4}$$

$$\mathbf{g}^{\mathrm{T}}: \quad \int_{\Omega^e} q^h (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \quad : \mathbf{g}_{\mathrm{V}}^{\mathrm{T}}, \tag{6.5}$$

$$\tilde{\mathbf{k}}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h) \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : \tilde{\mathbf{k}}_{\mathrm{V}}, \tag{6.6}$$

$$\tilde{\mathbf{c}}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h) \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_{\mathrm{V}}, \tag{6.7}$$

$$\tilde{\gamma}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h) \cdot \boldsymbol{\nabla} p^h d\Omega \quad : \tilde{\gamma}_{\mathrm{V}}, \tag{6.8}$$

$$\beta: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \beta_{\mathrm{V}}, \tag{6.9}$$

$$\gamma: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot (\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : \gamma_{\mathrm{V}}, \tag{6.10}$$

$$\theta: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \boldsymbol{\nabla} p^h d\Omega \quad : \theta_{\mathrm{V}}, \tag{6.11}$$

$$\mathbf{e}: \quad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) \rho (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \quad : \mathbf{e}_{\mathrm{V}}. \tag{6.12}$$

The element-level Reynolds and Courant numbers are defined the same way as they were defined before, as given by equations (4.6)-(4.8). The components of the element-matrix-based τ_{SUPG} are defined the same way as they were defined before, as given by equations (4)-(4.10). τ_{SUPG} is constructed from its components the same way as it was constructed before, as given by equation (4.11). The components of the element-vector-based τ_{SUPG} are defined the same way as they were defined before, as given by equations (4.12)-(4.14). The construction of $(\tau_{\mathrm{SUPG}})_{\mathrm{V}}$ is also the same as it was before, given by equation (4.15).
The components of the element-matrix-based τ_{PSPG} are defined as follows:

$$\tau_{\mathrm{P1}} = \frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma\|}, \tag{6.13}$$

$$\tau_{\mathrm{P2}} = \frac{\Delta t}{2} \frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\beta\|}, \tag{6.14}$$

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}} Re = \left(\frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma\|} \right) Re. \tag{6.15}$$

τ_{PSPG} is constructed from its components as follows:

$$\tau_{\mathrm{PSPG}} = \left(\frac{1}{\tau_{\mathrm{P1}}^r} + \frac{1}{\tau_{\mathrm{P2}}^r} + \frac{1}{\tau_{\mathrm{P3}}^r} \right)^{-\frac{1}{r}}. \tag{6.16}$$

The components of the element-vector-based τ_{PSPG} are defined as follows:

$$\tau_{\text{PV1}} = \tau_{\text{P1}}, \tag{6.17}$$

$$\tau_{\text{PV2}} = \tau_{\text{PV1}} \frac{\|\gamma_{\text{V}}\|}{\|\beta_{\text{V}}\|}, \tag{6.18}$$

$$\tau_{\text{PV3}} = \tau_{\text{PV1}} Re. \tag{6.19}$$

With these components,

$$(\tau_{\text{PSPG}})_{\text{V}} = \left(\frac{1}{\tau^r_{\text{PV1}}} + \frac{1}{\tau^r_{\text{PV2}}} + \frac{1}{\tau^r_{\text{PV3}}} \right)^{-\frac{1}{r}}. \tag{6.20}$$

The element-matrix-based τ_{LSIC} is defined as follows:

$$\tau_{\text{LSIC}} = \frac{\|\mathbf{c}\|}{\|\mathbf{e}\|}. \tag{6.21}$$

We define the element-vector-based τ_{LSIC} as:

$$(\tau_{\text{LSIC}})_{\text{V}} = \tau_{\text{LSIC}}. \tag{6.22}$$

7. UGN-Based Stabilization Parameters for Navier-Stokes Equations

For the purpose of comparison, we define here also the stabilization parameters that are based on an earlier definition of the length scale h [8]:

$$h_{\text{UGN}} = 2\, \|\mathbf{u}^h\| \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{7.1}$$

where N_a is the interpolation function associated with node a. The stabilization parameters are defined as follows:

$$\tau_{\text{SUGN1}} = \frac{h_{\text{UGN}}}{2\|\mathbf{u}^h\|}, \tag{7.2}$$

$$\tau_{\text{SUGN2}} = \frac{\Delta t}{2}, \tag{7.3}$$

$$\tau_{\text{SUGN3}} = \frac{h^2_{\text{UGN}}}{4\nu}, \tag{7.4}$$

$$(\tau_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{\tau^2_{\text{SUGN1}}} + \frac{1}{\tau^2_{\text{SUGN2}}} + \frac{1}{\tau^2_{\text{SUGN3}}} \right)^{-\frac{1}{2}}, \tag{7.5}$$

$$(\tau_{\text{PSPG}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}}, \tag{7.6}$$

$$(\tau_{\text{LSIC}})_{\text{UGN}} = \frac{h_{\text{UGN}}}{2} \|\mathbf{u}^h\|\, z. \tag{7.7}$$

Here z is given as follows:

$$z = \begin{cases} \left(\frac{Re_{\text{UGN}}}{3}\right) & Re_{\text{UGN}} \leq 3, \\ 1 & Re_{\text{UGN}} > 3, \end{cases} \tag{7.8}$$

where $Re_{\text{UGN}} = \frac{\|\mathbf{u}^h\| h_{\text{UGN}}}{2\nu}$.

Comparisons between the performances of these earlier stabilization parameters and the ones proposed here can be found in [11]. These comparisons show that, especially for special element geometries, the performances are similar.

It was pointed out in [13] that the expression for τ_{SUGN1} can be written more directly as

$$\tau_{\mathrm{SUGN1}} = \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{7.9}$$

and based on that, the expression for h_{UGN} can be written as

$$h_{\mathrm{UGN}} = 2\|\mathbf{u}^h\| \ \tau_{\mathrm{SUGN1}}. \tag{7.10}$$

A rationale for τ_{SUGN1} given by equation (7.9) was also provided in [13].

8. Discontinuity-Capturing Directional Dissipation (DCDD)

As a potential alternative or complement to the LSIC stabilization, we proposed in [12] the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization. In describing the DCDD stabilization, we first define the unit vectors $\mathbf{s}$ and $\mathbf{r}$:

$$\mathbf{s} = \frac{\mathbf{u}^h}{\|\mathbf{u}^h\|}, \ \mathbf{r} = \frac{\boldsymbol{\nabla}\|\mathbf{u}^h\|}{\|\ \boldsymbol{\nabla}\|\mathbf{u}^h\|\ \|}, \tag{8.1}$$

and the element-level matrices and vectors $\mathbf{c}_{\mathrm{r}}$, $\tilde{\mathbf{k}}_{\mathrm{r}}$, $(\mathbf{c}_{\mathrm{r}})_{\mathrm{V}}$, and $(\tilde{\mathbf{k}}_{\mathrm{r}})_{\mathrm{V}}$:

$$\mathbf{c}_{\mathrm{r}} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho(\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \qquad : (\mathbf{c}_{\mathrm{r}})_{\mathrm{V}} \ , \tag{8.2}$$

$$\tilde{\mathbf{k}}_{\mathrm{r}} : \quad \int_{\Omega^e} (\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{w}^h) \cdot \rho(\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : (\tilde{\mathbf{k}}_{\mathrm{r}})_{\mathrm{V}} \ . \tag{8.3}$$

Then the DCDD stabilization is defined as

$$S_{\mathrm{DCDD}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \rho \nu_{\mathrm{DCDD}} \boldsymbol{\nabla}\mathbf{w}^h : \left(\left[\mathbf{rr} - (\mathbf{r} \cdot \mathbf{s})^2 \mathbf{ss} \right] \cdot \boldsymbol{\nabla}\mathbf{u}^h \right) d\Omega, \tag{8.4}$$

where the element-matrix-based and element-vector-based DCDD viscosities are:

$$\nu_{\mathrm{DCDD}} \quad = \quad |\mathbf{r} \cdot \mathbf{u}^h| \frac{\|\mathbf{c}_{\mathrm{r}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \tag{8.5}$$

$$(\nu_{\mathrm{DCDD}})_{\mathrm{V}} \quad = \quad |\mathbf{r} \cdot \mathbf{u}^h| \frac{\|(\mathbf{c}_{\mathrm{r}})_{\mathrm{V}}\|}{\|(\tilde{\mathbf{k}}_{\mathrm{r}})_{\mathrm{V}}\|} \ . \tag{8.6}$$

An approximate version of the expression given by equation (8.5) can be written as

$$\nu_{\mathrm{DCDD}} = |\mathbf{r} \cdot \mathbf{u}^h| \frac{h_{\mathrm{RGN}}}{2}, \tag{8.7}$$

where

$$h_{\mathrm{RGN}} = 2 \left(\sum_{a=1}^{n_{en}} |\mathbf{r} \cdot \boldsymbol{\nabla} N_a| \right)^{-1}. \tag{8.8}$$

A different way of determining ν_{DCDD} can be expressed as

$$\nu_{\mathrm{DCDD}} = \tau_{\mathrm{DCDD}} \|\mathbf{u}^h\|^2, \tag{8.9}$$

where

$$\tau_{\mathrm{DCDD}} = \frac{h_{\mathrm{DCDD}}}{2\|\mathbf{U}\|} \, \frac{\| \boldsymbol{\nabla}\|\mathbf{u}^h\| \, \| \, h_{\mathrm{DCDD}}}{\|\mathbf{U}\|} \ . \tag{8.10}$$

Here $\mathbf{U}$ represents a global velocity scale, and h_{DCDD} can be calculated by using the expression

$$h_{\mathrm{DCDD}} = 2 \, \frac{\|\mathbf{c}_{\mathrm{r}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \tag{8.11}$$

or the approximation

$$h_{\mathrm{DCDD}} = h_{\mathrm{RGN}} \ . \tag{8.12}$$

Combining equations (8.9) and (8.10), we obtain

$$\nu_{\mathrm{DCDD}} = \frac{1}{2} \left(\frac{\|\mathbf{u}^h\|}{\|\mathbf{U}\|} \right)^2 (h_{\mathrm{DCDD}})^2 \, \| \, \boldsymbol{\nabla}\|\mathbf{u}^h\| \, \| \ . \tag{8.13}$$

9. UGN/RGN-Based Stabilization Parameters for Navier-Stokes Equations

In [4], we proposed to re-define τ_{PSPG} and provided the reason for doing that. We described how we re-define τ_{PSPG} by modifying the definitions of τ_{P3} and τ_{PV3} given by equations (6.15) and (6.19). We proposed to accomplish that by using the expressions

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \ \tau_{\mathrm{PV3}} = \tau_{\mathrm{PV1}} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \tag{9.1}$$

or the approximations

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}} \, Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 \ , \ \tau_{\mathrm{PV3}} = \tau_{\mathrm{PV1}} \, Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 \ . \tag{9.2}$$

In [4], we further stated that these modifications can also be applied to τ_{S3} and τ_{SV3} given by equations (4.10) and (4.14). In [13], we wrote those expressions explicitly as follows:

$$\tau_{\mathrm{S3}} = \tau_{\mathrm{S1}} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \ \tau_{\mathrm{SV3}} = \tau_{\mathrm{SV1}} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} \ , \tag{9.3}$$

$$\tau_{\mathrm{S3}} = \tau_{\mathrm{S1}} \, Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 \ , \ \tau_{\mathrm{SV3}} = \tau_{\mathrm{SV1}} \, Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 \ . \tag{9.4}$$

We noted in [13] that if we are dealing with just an advection-diffusion equation, rather than the Navier-Stokes equations of incompressible flows, then the definition of the unit vector $\mathbf{r}$ changes as follows:

$$\mathbf{r} = \frac{\boldsymbol{\nabla}|\phi^h|}{\| \, \boldsymbol{\nabla}|\phi^h| \, \|} \ . \tag{9.5}$$

We also proposed in [13] to re-define τ_{SUGN3} given by equation (7.4) as follows:

$$\tau_{\mathrm{SUGN3}} = \frac{h_{\mathrm{RGN}}^2}{4\nu} \; . \tag{9.6}$$

Furthermore, we proposed in [13] to replace $(\tau_{\mathrm{LSIC}})_{\mathrm{UGN}}$ given by equation (7.7) as follows:

$$(\tau_{\mathrm{LSIC}})_{\mathrm{UGN}} = (\tau_{\mathrm{SUPG}})_{\mathrm{UGN}} \left\|\mathbf{u}^h\right\|^2 \; . \tag{9.7}$$

We further commented in [13] that the "element length"s h_{UGN} (given by equation (7.1)) and h_{RGN} (equation (8.8)) can be viewed as the local length scales corresponding to the advection- and diffusion-dominated limits, respectively.

10. Deforming-Spatial-Domain/Stabilized Space-Time (DSD/SST) Formulation

In the DSD/SST method, the finite element formulation of the governing equations is written over a sequence of N space-time slabs Q_n, where Q_n is the slice of the space-time domain between the time levels t_n and t_{n+1}. At each time step, the integrations involved in the finite element formulation are performed over Q_n. The space-time finite element interpolation functions are continuous within a space-time slab, but discontinuous from one space-time slab to another. Typically we use first-order polynomials as interpolation functions. The notation $(\cdot)_n^-$ and $(\cdot)_n^+$ denotes the function values at t_n as approached from below and above, respectively. Each Q_n is decomposed into space-time elements Q_n^e, where $e = 1, 2, \ldots, (n_{el})_n$. The subscript n used with n_{el} is to account for the general case in which the number of space-time elements may change from one space-time slab to another. The Dirichlet- and Neumann-type boundary conditions are enforced over $(P_n)_{\mathrm{g}}$ and $(P_n)_{\mathrm{h}}$, the complementary subsets of the lateral boundary of the space-time slab. The finite element trial function spaces $(\mathcal{S}_{\mathbf{u}}^h)_n$ for velocity and $(\mathcal{S}_p^h)_n$ for pressure, and the test function spaces $(\mathcal{V}_{\mathbf{u}}^h)_n$ and $(\mathcal{V}_p^h)_n = (\mathcal{S}_p^h)_n$ are defined by using, over Q_n, first-order polynomials in both space and time.

The DSD/SST formulation is written as follows: given $(\mathbf{u}^h)_n^-$, find $\mathbf{u}^h \in (\mathcal{S}_{\mathbf{u}}^h)_n$ and $p^h \in (\mathcal{S}_p^h)_n$ such that $\forall \mathbf{w}^h \in (\mathcal{V}_{\mathbf{u}}^h)_n$ and $q^h \in (\mathcal{V}_p^h)_n$:

$$\begin{aligned}
& \int_{Q_n} \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}^h \right) dQ + \int_{Q_n} \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) dQ \\
- & \int_{(P_n)_{\mathrm{h}}} \mathbf{w}^h \cdot \mathbf{h}^h dP + \int_{Q_n} q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ + \int_{\Omega_n} (\mathbf{w}^h)_n^+ \cdot \rho \left((\mathbf{u}^h)_n^+ - (\mathbf{u}^h)_n^- \right) d\Omega \\
+ & \sum_{e=1}^{(n_{el})_n} \int_{Q_n^e} \frac{\tau_{\mathrm{LSME}}}{\rho} Ł(q^h, \mathbf{w}^h) \cdot \left[Ł(p^h, \mathbf{u}^h) - \rho \mathbf{f}^h \right] dQ \\
+ & \sum_{e=1}^{n_{el}} \int_{Q_n^e} \tau_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ = 0,
\end{aligned} \tag{10.1}$$

where

$$Ł(q^h, \mathbf{w}^h) = \rho \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(q^h, \mathbf{w}^h), \tag{10.2}$$

and τ_{LSME} and τ_{LSIC} are the stabilization parameters (see [14]). This formulation is applied to all space-time slabs $Q_0, Q_1, Q_2, \ldots, Q_{N-1}$, starting with $(\mathbf{u}^h)_0^- = \mathbf{u}_0$. For an earlier, detailed reference on this stabilized formulation see [1].

In [13] we wrote a DSD/SST formulation that was slightly different than the one given by equation (10.1). We did that by neglecting the $(\tau_{\mathrm{LSME}}/\rho)\, \boldsymbol{\nabla} \cdot (2\mu \boldsymbol{\varepsilon}(\mathbf{w}^h))$ term and replacing τ_{LSME} with τ_{SUPG} and τ_{PSPG}:

$$\begin{aligned}
& \int_{Q_n} \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}^h \right) dQ + \int_{Q_n} \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) dQ \\
- & \int_{(P_n)_{\mathrm{h}}} \mathbf{w}^h \cdot \mathbf{h}^h dP + \int_{Q_n} q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ + \int_{\Omega_n} (\mathbf{w}^h)_n^+ \cdot \rho \left((\mathbf{u}^h)_n^+ - (\mathbf{u}^h)_n^- \right) d\Omega \\
+ & \sum_{e=1}^{(n_{el})_n} \int_{Q_n^e} \frac{1}{\rho} \left[\tau_{\mathrm{SUPG}} \rho \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \nabla \mathbf{w}^h \right) + \tau_{\mathrm{PSPG}} \nabla q^h \right] \cdot \left[Ł(p^h, \mathbf{u}^h) - \rho \mathbf{f}^h \right] dQ \\
+ & \sum_{e=1}^{n_{el}} \int_{Q_n^e} \tau_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ = 0.
\end{aligned} \tag{10.3}$$

11. Element-Matrix-Based Stabilization Parameters for the DSD/SST Formulation

For extensions of the τ calculations based on matrix norms to the DSD/SST formulation, in [13] we defined the space-time augmented versions of the element-level matrices and vectors given by equations (6.2), (6.6), and (6.10) as follows:

$$\mathbf{c}_{\mathrm{A}} : \quad \int_{Q_n^e} \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\mathbf{c}_{\mathrm{A}})_{\mathrm{V}}, \tag{11.1}$$

$$\tilde{\mathbf{k}}_{\mathrm{A}} : \quad \int_{Q_n^e} \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h \right) \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\tilde{\mathbf{k}}_{\mathrm{A}})_{\mathrm{V}}, \tag{11.2}$$

$$\gamma_{\mathrm{A}} : \quad \int_{Q_n^e} \boldsymbol{\nabla} q^h \cdot \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\gamma_{\mathrm{A}})_{\mathrm{V}}. \tag{11.3}$$

The components of element-matrix-based τ_{SUPG} were defined in [13] as follows:

$$\tau_{\mathrm{S12}} = \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{A}}\|}, \tag{11.4}$$

$$\tau_{\mathrm{S3}} = \tau_{\mathrm{S12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu\, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}, \tag{11.5}$$

where $\tilde{\mathbf{k}}_{\mathrm{r}}$ is the space-time version (i.e. integrated over the space-time element domain Q_n^e) of the element-level matrix given by equation (8.3). To construct τ_{SUPG} from its components we proposed in [13] the form

$$\tau_{\mathrm{SUPG}} = \left(\frac{1}{\tau_{\mathrm{S12}}^r} + \frac{1}{\tau_{\mathrm{S3}}^r} \right)^{-\frac{1}{r}} . \tag{11.6}$$

The components of the element-vector-based τ_{SUPG} were defined in [13] as

$$\tau_{\mathrm{SV12}} = \frac{\|(\mathbf{c}_{\mathrm{A}})_{\mathrm{V}}\|}{\|(\tilde{\mathbf{k}}_{\mathrm{A}})_{\mathrm{V}}\|}, \tag{11.7}$$

$$\tau_{\mathrm{SV3}} = \tau_{\mathrm{SV12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}. \tag{11.8}$$

From these two components,

$$(\tau_{\mathrm{SUPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{SV12}}^r} + \frac{1}{\tau_{\mathrm{SV3}}^r} \right)^{-\frac{1}{r}} . \tag{11.9}$$

The components of element-matrix-based τ_{PSPG} were defined in [13] as follows:

$$\tau_{\mathrm{P12}} = \frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma_{\mathrm{A}}\|}, \tag{11.10}$$

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}, \tag{11.11}$$

where $\mathbf{g}^{\mathrm{T}}$ is the space-time version of the element-level matrix given by equation (6.5). To construct τ_{PSPG} from its components, we proposed in [13] the form

$$\tau_{\mathrm{PSPG}} = \left(\frac{1}{\tau_{\mathrm{P12}}^r} + \frac{1}{\tau_{\mathrm{P3}}^r} \right)^{-\frac{1}{r}} . \tag{11.12}$$

The components of the element-vector-based τ_{PSPG} were defined in [13] as follows:

$$\tau_{\mathrm{PV12}} = \frac{\|\mathbf{g}_{\mathrm{V}}^{\mathrm{T}}\|}{\|(\gamma_{\mathrm{A}})_{\mathrm{V}}\|}, \tag{11.13}$$

$$\tau_{\mathrm{PV3}} = \tau_{\mathrm{PV12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}. \tag{11.14}$$

From these components,

$$(\tau_{\mathrm{PSPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{PV12}}^r} + \frac{1}{\tau_{\mathrm{PV3}}^r} \right)^{-\frac{1}{r}} . \tag{11.15}$$

The element-matrix-based τ_{LSIC} was defined in [13] as

$$\tau_{\mathrm{LSIC}} = \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\|\mathbf{e}\|}, \tag{11.16}$$

where $\mathbf{e}$ is the space-time version of the element-level matrix given by equation (6.12).

The element-vector-based τ_{LSIC} was defined in [13] as

$$(\tau_{\text{LSIC}})_{\text{V}} = \tau_{\text{LSIC}}. \tag{11.17}$$

12. UGN/RGN-Based Stabilization Parameters for the DSD/SST Formulation

The space-time versions of τ_{SUGN1}, τ_{SUGN2}, τ_{SUGN3}, $(\tau_{\text{SUPG}})_{\text{UGN}}$, $(\tau_{\text{PSPG}})_{\text{UGN}}$, and $(\tau_{\text{LSIC}})_{\text{UGN}}$, given respectively by Eqs. (7.2), (7.3), (9.6), (7.5), (7.6), and (9.7), were defined in [13] as follows:

$$\tau_{\text{SUGN12}} = \left(\sum_{a=1}^{n_{en}} \left| \frac{\partial N_a}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} N_a \right| \right)^{-1}, \tag{12.1}$$

$$\tau_{\text{SUGN3}} = \frac{h_{\text{RGN}}^2}{4\nu}, \tag{12.2}$$

$$(\tau_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{\tau_{\text{SUGN13}}^2} + \frac{1}{\tau_{\text{SUGN3}}^2} \right)^{-\frac{1}{2}}, \tag{12.3}$$

$$(\tau_{\text{PSPG}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}}, \tag{12.4}$$

$$(\tau_{\text{LSIC}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}} \left\| \mathbf{u}^h \right\|^2. \tag{12.5}$$

Here, n_{en} is the number of nodes for the space-time element, and N_a is the space-time interpolation function associated with node a.

13. Test Computations

13.1. 1D Advection of a Cosine Wave.

13.1. 1D Advection of a Cosine Wave. In this test, we compute the advection of a cosine wave with the space-time SUPG formulation at dimensionless wave number, $q = kh = 0.3142$ and Courant number, $Cr_u = 1.0$ and 0.5. Figure 1 shows the space-time mesh for $Cr_u = 1.0$ and 0.5.

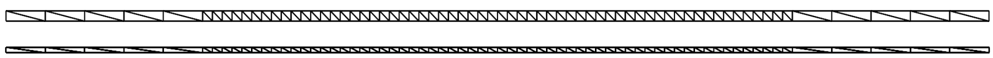

Figure 1. 1D advection of a cosine wave. Meshes for space-time computations. Courant number, $Cr_u = 1.0$ (top) and 0.5 (bottom).

We compare the solutions obtained with the semi-discrete and space-time versions of τ. For the semi-discrete version of τ, we use the expressions given by equations (7.9), (7.3) and (7.5), without τ_{SUGN3}. For the space-time version of τ, we use the expression given by equation (12.1). Figure 2 shows that, for both Courant numbers, the solution obtained with the semi-discrete and space-time versions of τ are very close.

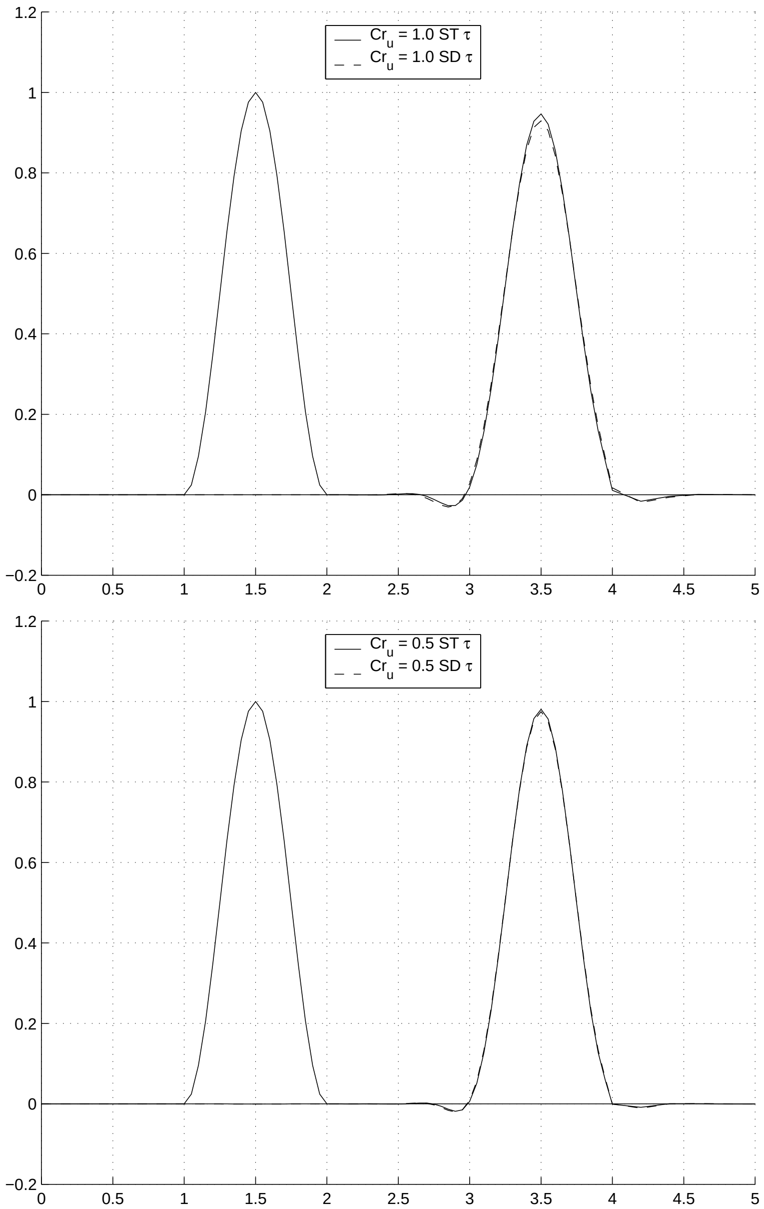

Figure 2. 1D advection of a cosine wave. Stabilized space-time computations with the semi-discrete and space-time versions of τ. Dimensionless wave number, $q = kh = 0.3142$. Courant number, $Cr_u = 1.0$ (top) and 0.5 (bottom).

13.2. **2D Incompressible Flow Past a Cylinder at Re = 100.** In this test computation, for meshes containing elements with high aspect ratios, we evaluate the performance of the SUPG/PSPG formulation with UGN/RGN-based stabilization parameters. The test problem we use, 2D incompressible flow past a cylinder at Re = 100, is a well-studied problem, with an easily identifiable Karman vortex shedding (see Figure 3).

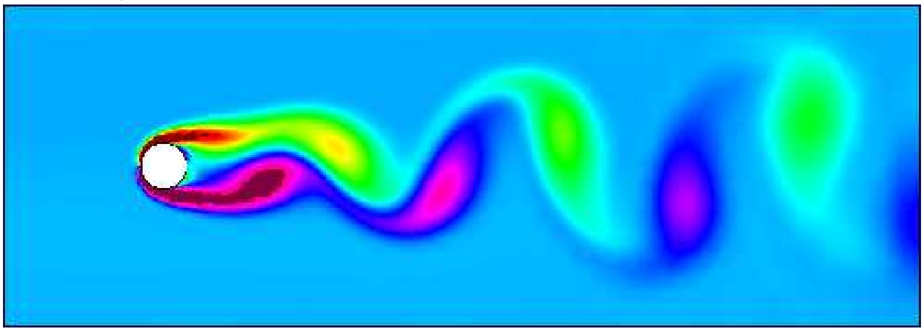

Figure 3. 2D Incompressible Flow Past a Cylinder at Re = 100. Computed with SUPG/PSPG formulation with UGN/RGN-based stabilization parameters. Vorticity.

Figure 4 (4a,b and c) shows the triangular mesh in the boundary layer, the velocity vectors near the cylinder, and the velocity vectors in the boundary layer. Although the aspect ratio of the elements adjacent to the cylinder surface is 100, the SUPG/PSPG formulation with the UGN/RGN-based stabilization parameters performs very well.

14. **Concluding Remarks**

We described how we determine the stabilization parameters ("τ"s) and element length scales used in stabilized finite element formulations of flow problems. These stabilized formulations include the interface-tracking and interface-capturing techniques we developed for computation of flows with moving boundaries and interfaces. The interface-tracking techniques are based on the Deforming-Spatial-Domain/Stabilized Space-Time formulation, where the mesh moves to track the interface. The interface-capturing techniques, typically used with non-moving meshes, are based on a stabilized semi-discrete formulation of the Navier-Stokes equations, combined with a stabilized formulation of an advection equation. The advection equation governs the time-evolution of an interface function marking the interface location. As specific stabilization methods, we focused on the streamline-upwind/Petrov-Galerkin (SUPG) and pressure-stabilizing/Petrov-Galerkin (PSPG) methods. For the Navier-Stokes equations and the advection equation, we described the element-matrix-based and element-vector-based τs designed for semi-discrete and space-time formulations. These τ definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They take into account the local length scales, advection field and the element-level Reynolds number. Based on these definitions, a τ can be calculated

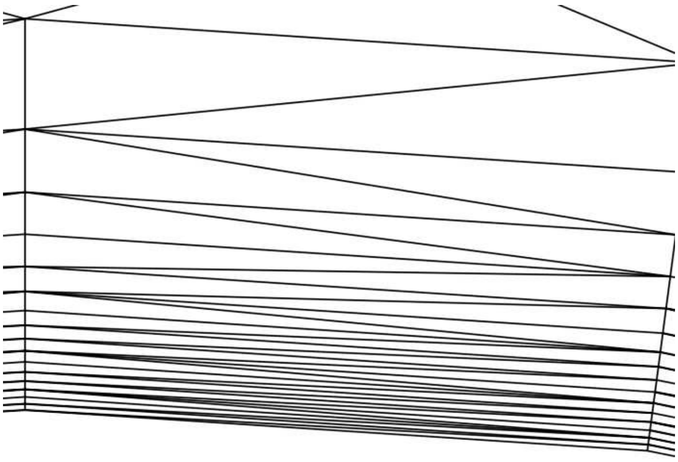

Figure 4a

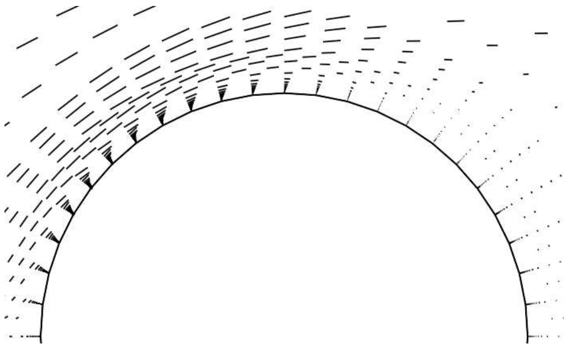

Figure 4b

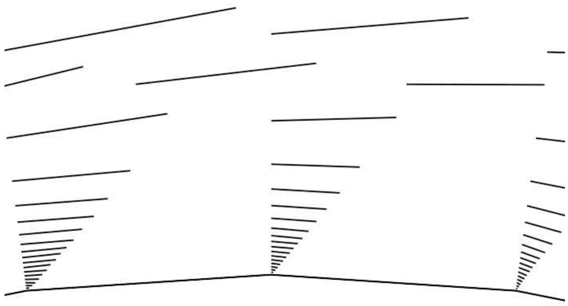

Figure 4c

Figure 4. 2D Incompressible Flow Past a Cylinder at Re = 100. Computed with SUPG/PSPG formulation with UGN/RGN-based stabilization parameters. Top: mesh in the boundary layer; Middle: velocity vectors near the cylinder; Bottom: velocity vectors in the boundary layer.

for each element, or even for each element node or degree of freedom or element equation. We also described certain variations and complements of these new τs, including the approximate versions that are based on the local length scales for the advection- and diffusion-dominated limits. With test problems for the advection-diffusion and Navier-Stokes equations, we showed that the stabilization parameters described perform well, even for elements with high aspect ratios.

REFERENCES

1. Tezduyar, T.: *Stabilized finite element formulations for incompressible flow computations*, Advances in Applied Mechanics, **28**, (1991), 1–44.
2. Tezduyar, T.: *Finite element methods for flow problems with moving boundaries and interfaces*, Archives of Computational Methods in Engineering, **8**, (2001), 83–130.
3. Tezduyar, T.: *Finite element interface-tracking and interface-capturing techniques for flows with moving boundaries and interfaces*, in *Proceedings of the ASME Symposium on Fluid-Physics and Heat Transfer for Macro- and Micro-Scale Gas-Liquid and Phase-Change Flows (CD-ROM)*, ASME Paper IMECE2001/HTD-24206, ASME, New York, New York, 2001.
4. Tezduyar, T.: *Stabilized finite element formulations and interface-tracking and interface-capturing techniques for incompressible flows* (2002), to appear in *Proceedings of the Workshop on Numerical Simulations of Incompressible Flows*, Half Moon Bay, California.
5. Hughes, T. and Brooks, A.: *A multi-dimensional upwind scheme with no crosswind diffusion*, in T. Hughes, ed., *Finite Element Methods for Convection Dominated Flows*, AMD-Vol.34, ASME, New York (1979), 19–35.
6. Hughes, T., Franca, L. and Hulbert, G.: *A new finite element formulation for computational fluid dynamics: VIII. the Galerkin/least-squares method for advective-diffusive equations*, Computer Methods in Applied Mechanics and Engineering, **73**, (1989), 173–189.
7. Tezduyar, T. and Hughes, T.: *Finite element formulations for convection dominated flows with particular emphasis on the compressible Euler equations*, in *Proceedings of AIAA 21st Aerospace Sciences Meeting*, AIAA Paper 83-0125, Reno, Nevada, 1983.
8. Tezduyar, T. and Park, Y.: *Discontinuity capturing finite element formulations for nonlinear convection-diffusion-reaction problems*, Computer Methods in Applied Mechanics and Engineering, **59**, (1986), 307–325.
9. Tezduyar, T. and Ganjoo, D.: *Petrov-Galerkin formulations with weighting functions dependent upon spatial and temporal discretization: Applications to transient convection-diffusuion problmes*, Computer Methods in Applied Mechanics and Engineering, **59**, (1986), 49–71.
10. L. Franca, L., Frey, S. and Hughes, T.: *Stabilized finite element methods: I. Application to the advective-diffusive model*, Computer Methods in Applied Mechanics and Engineering, **95**, (1992), 253–276.
11. Tezduyar, T. and Osawa, Y.: *Finite element stabilization parameters computed from element matrices and vectors*, Computer Methods in Applied Mechanics and Engineering, **190**, (2000), 411–430.

12. Tezduyar, T.: *Adaptive determination of the finite element stabilization parameters*, in *Proceedings of the ECCOMAS Computational Fluid Dynamics Conference 2001 (CD-ROM)*, Swansea, Wales, United Kingdom, 2001.
13. Tezduyar, T.: *Stabilization parameters and local length scales in SUPG and PSPG formulations* (2002), to appear in *Proceedings of the Fifth World Congress on Computational Mechanics* (Web Site), Vienna, Austria.
14. Tezduyar, T. and Osawa, Y.: *Methods for parallel computation of complex flow problems*, Parallel Computing, **25**, (1999), 2039–2066.